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Formulas for Higher Derivatives of the Riemann Zeta Function

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Abstract. The functional equation for $\zeta(s)$ is used to obtain formulas for all derivatives $\zeta^{(k)}(s)$. A closed form evaluation of $\zeta^{(k)}(0)$ is given, and numerical values are computed to 15D for $k = 0(1)18$.

The functional equation for the Riemann zeta function states that

$$\zeta(1-s) = 2(2\pi)^{-s}\cos\frac{\pi s}{2}\Gamma(s)\zeta(s) \tag{1}$$

(see [3, Theorem 12.7]). If this is differentiated k times we obtain a formula which, as noted by Spira [11], can be put in the form

$$(-1)^k\zeta^{(k)}(1-s) = 2(2\pi)^{-s}\sum_{j=0}^{k}\sum_{m=0}^{k}\left(a_{jkm}\cos\frac{\pi s}{2} + b_{jkm}\sin\frac{\pi s}{2}\right)\Gamma^{(j)}(s)\zeta^{(m)}(s),$$

where the coefficients a_{jkm} and b_{jkm} are independent of s. This formula was used by Spira [11], [12] to determine zero-free regions for $\zeta^{(k)}(s)$, and by Berndt [5], to determine the asymptotic number of zeros of $\zeta^{(k)}(s)$ with $0 < t < T$, where $s = \sigma + it$.

This paper gives a variant of this formula (Theorem 1) which enables us to determine the coefficients a_{jkm} and b_{jkm} explicitly (Theorem 2). Our version also leads to a closed form evaluation of $\zeta^{(k)}(0)$ (Theorem 3) which contains the well-known values $\zeta(0) = -\frac{1}{2}$ and $\zeta'(0) = -\frac{1}{2}\log(2\pi)$ as well as a formula for $\zeta''(0)$ obtained by Ramanujan. The results for $k \geqslant 3$ appear to be new. Alternate formulas expressing $\zeta^{(k)}(s)$ in terms of integrals are also given (Theorem 4). The values of $\zeta^{(k)}(0)$ are computed to 16S for $k = 0(1)18$ (Table 2).

Notation. Throughout this paper, z denotes the fixed complex number $x + iy$ with $x = -\log 2\pi$, $y = -\pi/2$, and z^* denotes the complex conjugate of z.

THEOREM 1. *For each integer $k \geqslant 1$ and all complex s we have*

$$(-1)^k\zeta^{(k)}(1-s) = \sum_{m=0}^{k}\binom{k}{m}\left\{e^{sz}z^{k-m} + e^{sz^*}(z^*)^{k-m}\right\}\left\{\Gamma(s)\zeta(s)\right\}^{(m)}. \tag{2}$$

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Proof. First we put the functional equation in a form which is more convenient for repeated differentiation. Since $(2\pi)^{-s} = e^{-s\log(2\pi)}$ and $2\cos \pi s/2 = e^{\pi i s/2} + e^{-\pi i s/2}$ we can rewrite (1) as follows:

$$\zeta(1-s) = \varphi(s; z) + \varphi(s; z^*), \tag{3}$$

where

$$\varphi(s; z) = \Gamma(s)\zeta(s)e^{sz}.$$

Differentiation of (3) gives us

$$(-1)^k\zeta^{(k)}(1-s) = \varphi^{(k)}(s; z) + \varphi^{(k)}(s; z^*). \tag{4}$$

Using Leibniz's rule to calculate $\varphi^{(k)}(s; z)$ we find

$$\varphi^{(k)}(s; z) = e^{sz}\sum_{m=0}^{k}\binom{k}{m}z^{k-m}\{\Gamma(s)\zeta(s)\}^{(m)}$$

which, together with (4), proves (2).

THEOREM 2. *For each integer $k \geqslant 1$ and all complex s we have*

$$\begin{aligned} (-1)^k\zeta^{(k)}(1-s) &= 2(2\pi)^{-s}\sum_{m=0}^{k}\binom{k}{m}\Big\{\operatorname{Re}(z^{k-m})\cos\frac{\pi s}{2} \\ &\qquad + \operatorname{Im}(z^{k-m})\sin\frac{\pi s}{2}\Big\}\{\Gamma(s)\zeta(s)\}^{(m)} \qquad (5) \\ &= 2(2\pi)^{-s}\sum_{m=0}^{k}\sum_{r=0}^{m}\binom{k}{m}\binom{m}{r}\Big\{\operatorname{Re}(z^{k-m})\cos\frac{\pi s}{2} \\ &\qquad + \operatorname{Im}(z^{k-m})\sin\frac{\pi s}{2}\Big\}\Gamma^{(r)}(s)\zeta^{(m-r)}(s). \qquad (6) \end{aligned}$$

Proof. To deduce (5) from (2) we note that

$$e^{sz}z^{k-m} + e^{sz^*}(z^*)^{k-m} = 2(2\pi)^{-s}\Big\{\operatorname{Re}(z^{k-m})\cos\frac{\pi s}{2} + \operatorname{Im}(z^{k-m})\sin\frac{\pi s}{2}\Big\},$$

and to deduce (6) from (5) we use Leibniz's rule for the mth derivative $\{\Gamma(s)\zeta(s)\}^{(m)}$.

Examples. If $z = x + iy$ we have

$$\operatorname{Re}(z^2) = x^2 - y^2, \qquad \operatorname{Im}(z^2) = 2xy,$$
$$\operatorname{Re}(z^3) = x^3 - 3xy^2, \qquad \operatorname{Im}(z^3) = 3x^2y - y^3.$$

When $x = -\log 2\pi$ and $y = -\pi/2$ we find, by taking $k = 1, 2, 3$ in (5),

$$\begin{aligned} -\zeta'(1-s) &= 2(2\pi)^{-s}\Big\{x\cos\frac{\pi s}{2} + y\sin\frac{\pi s}{2}\Big\}\Gamma(s)\zeta(s) \\ &\quad + 2(2\pi)^{-s}\cos\frac{\pi s}{2}\{\Gamma(s)\zeta(s)\}', \\ \zeta''(1-s) &= 2(2\pi)^{-s}\Big\{(x^2-y^2)\cos\frac{\pi s}{2} + 2xy\sin\frac{\pi s}{2}\Big\}\Gamma(s)\zeta(s) \\ &\quad + 2(2\pi)^{-s}\Big\{2x\cos\frac{\pi s}{2} + 2y\sin\frac{\pi s}{2}\Big\}\{\Gamma(s)\zeta(s)\}' \\ &\quad + 2(2\pi)^{-s}\cos\frac{\pi s}{2}\{\Gamma(s)\zeta(s)\}'', \end{aligned}$$

$$-\zeta'''(1-s) = 2(2\pi)^{-s}\Big\{(x^3-3xy^2)\cos\frac{\pi s}{2} + (3x^2y - y^3)\sin\frac{\pi s}{2}\Big\}\Gamma(s)\zeta(s)$$
$$+2(2\pi)^{-s}\Big\{3(x^2-y^2)\cos\frac{\pi s}{2} + 6xy\sin\frac{\pi s}{2}\Big\}\{\Gamma(s)\zeta(s)\}'$$
$$+2(2\pi)^{-s}\Big\{3x\cos\frac{\pi s}{2} + 3y\sin\frac{\pi s}{2}\Big\}\{\Gamma(s)\zeta(s)\}''$$
$$+2(2\pi)^{-s}\cos\frac{\pi s}{2}\{\Gamma(s)\zeta(s)\}'''.$$

It should be noted that when s is an integer one of the factors $\cos \pi s/2$ or $\sin \pi s/2$ vanishes, and Eqs. (5) and (6) simplify further. For example, if $s = 2n+1$, where $n = 1, 2, 3, \ldots$, we have $\cos \pi s/2 = 0$ and $\sin \pi s/2 = (-1)^n$ and (6) becomes

$$(-1)^k\zeta^{(k)}(-2n)$$
$$= \frac{2(-1)^n}{(2\pi)^{2n+1}}\sum_{m=0}^{k}\sum_{r=0}^{m}\binom{k}{m}\binom{m}{r}\operatorname{Im}(z^{k-m})\Gamma^{(r)}(2n+1)\zeta^{(m-r)}(2n+1).$$

Thus, $\zeta^{(k)}(-2n)$ is a linear combination of $\zeta(2n+1), \zeta'(2n+1), \ldots, \zeta^{(k)}(2n+1)$.

Similarly, when $s = 2n$ the sine terms vanish and we get

$$(-1)^k\zeta^{(k)}(1-2n) = \frac{2(-1)^n}{(2\pi)^{2n}}\sum_{m=0}^{k}\sum_{r=0}^{m}\binom{k}{m}\binom{m}{r}\operatorname{Re}(z^{k-m})\Gamma^{(r)}(2n)\zeta^{(m-r)}(2n),$$

a linear combination of $\zeta(2n), \zeta'(2n), \ldots, \zeta^{(k)}(2n)$.

If we put $s = 1$ in (2), we get $(-1)^k\zeta^{(k)}(0)$ on the left, but on the right we have an indeterminate form. By expanding each of the functions e^{sz}, e^{sz^*} and $\{\Gamma(s)\zeta(s)\}^{(m)}$ in powers of $s-1$ and letting $s \to 1$ we can obtain a closed form for $(-1)^k\zeta^{(k)}(0)$. A simpler method which gives the same result is based on the functional equation in (1).

Since the left member of (1) is analytic at $s = 1$ it has a power series expansion

$$\zeta(1-s) = \sum_{n=0}^{\infty}\frac{(-1)^n\zeta^{(n)}(0)}{n!}(s-1)^n.$$

Now we expand the right member of (1) in powers of $s-1$ and equate coefficients. Again we use Eq. (3) which served us so well in proving Theorem 1, and first find the expansion of $\varphi(s; z)$ in powers of $s-1$.

The product $\Gamma(s)\zeta(s)$ has a Laurent expansion of the form

$$\Gamma(s)\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty}a_n(s-1)^n, \tag{7}$$

and for the exponential factor e^{sz} we write

$$e^{sz} = e^z e^{(s-1)z} = \sum_{n=0}^{\infty}e_n(z)(s-1)^n,$$

where

$$e_n(z) = \frac{e^z z^n}{n!}.$$

Therefore the product $\varphi(s; z) = \Gamma(s)\zeta(s)e^{sz}$ has the expansion

$$\varphi(s; z) = \left(\frac{1}{s-1} + \sum_{n=0}^{\infty} a_n (s-1)^n\right)\left(\sum_{n=0}^{\infty} e_n(z)(s-1)^n\right)$$
$$= \frac{e^z}{s-1} + \sum_{n=0}^{\infty}\left(e_{n+1}(z) + \sum_{k=0}^{n} a_k e_{n-k}(z)\right)(s-1)^n.$$

Equating coefficients of $(s-1)^n$ in the functional equation (3) we find, for $n \geqslant 0$,

$$(8) \qquad (-1)^n \frac{\zeta^{(n)}(0)}{n!} = e_{n+1}(z) + e_{n+1}(z^*) + \sum_{k=0}^{n} a_k(e_{n-k}(z) + e_{n-k}(z^*)).$$

But $e^z = -i/(2\pi)$ and $e^{z^*} = i/(2\pi)$, so

$$e_n(z) + e_n(z^*) = \frac{iz^n - i(z^*)^n}{2\pi n!} = \frac{1}{\pi}\frac{\operatorname{Im}(z^n)}{n!}.$$

Hence (8) becomes

$$(-1)^n \frac{\zeta^{(n)}(0)}{n!} = \frac{1}{\pi}\frac{\operatorname{Im}(z^{n+1})}{(n+1)!} + \frac{1}{\pi}\sum_{k=0}^{n} a_k \frac{\operatorname{Im}(z^{n-k})}{(n-k)!}.$$

Since $\operatorname{Im}(z^0) = 0$ and $a_0 = 0$ (as we will show later), the first and last terms of the sum can be deleted and we obtain the following theorem.

THEOREM 3. *If* $z = -\log(2\pi) - i\pi/2$ *and* $n \geqslant 0$, *we have*

$$(9) \qquad (-1)^n \frac{\zeta^{(n)}(0)}{n!} = \frac{1}{\pi}\frac{\operatorname{Im}(z^{n+1})}{(n+1)!} + \frac{1}{\pi}\sum_{k=1}^{n-1} a_k \frac{\operatorname{Im}(z^{n-k})}{(n-k)!},$$

where the coefficients a_k *are determined by* (7).

Examples. For $0 \leqslant n \leqslant 4$, we find that (9) gives us

$$\zeta(0) = -\frac{1}{2},$$

$$\zeta'(0) = -\frac{1}{2\pi}\operatorname{Im}(z^2) = -\frac{xy}{\pi} = -\frac{1}{2}\log(2\pi),$$

$$\zeta''(0) = \frac{1}{3\pi}\operatorname{Im}(z^3) + \frac{2}{\pi}a_1\operatorname{Im}(z) = \frac{1}{3\pi}(3x^2y - y^3) + \frac{2}{\pi}a_1 y$$
$$= -\frac{1}{2}\log^2(2\pi) + \frac{\pi^2}{24} - a_1,$$

$$\zeta'''(0) = -\frac{1}{4\pi}\operatorname{Im}(z^4) - \frac{3!}{\pi}\sum_{k=1}^{2} a_k \frac{\operatorname{Im}(z^{3-k})}{(3-k)!}$$
$$= -\frac{1}{2}\log^3(2\pi) + \frac{\pi^2}{8}\log(2\pi) - 3a_1\log(2\pi) + 3a_2,$$

$$\zeta^{(4)}(0) = \frac{1}{5\pi}\operatorname{Im}(z^5) + \frac{4!}{\pi}\sum_{k=1}^{3} a_k \frac{\operatorname{Im}(z^{4-k})}{(4-k)!}$$
$$= -\frac{1}{2}\log^4(2\pi) + \frac{\pi^2}{4}\log^2(2\pi) - \frac{\pi^4}{160} - 6a_1\log^2(2\pi)$$
$$+ \frac{\pi^2}{2}a_1 + 12a_2\log(2\pi) - 12a_3.$$

The formulas for $\zeta(0)$ and $\zeta'(0)$ are well-known [13, p. 20], and the formula for $\zeta''(0)$ was obtained by Ramanujan [6, p. 25]. Numerical values are given below in Table 2.

The coefficients a_k which appear in (9) and are defined by (7) can be calculated. They are related to the coefficients in the Laurent expansion

$$\zeta(s+1) = \frac{1}{s} + \sum_{n=0}^{\infty} A_n s^n \tag{10}$$

and those in the power series expansion

$$\Gamma(s+1) = \sum_{n=0}^{\infty} c_n s^n. \tag{11}$$

The A_n are named after Stieltjes who showed [4, p. 155] that

$$(-1)^n n! A_n = \lim_{N\to\infty}\left(\sum_{k=1}^{N} \frac{\log^n k}{k} - \frac{\log^{n+1} N}{n+1}\right).$$

In particular, A_0 is Euler's constant γ. The first 20 Stieltjes constants have been calculated by Liang and Todd [9].

The numbers c_n in (11) are, of course, $\Gamma^{(n)}(1)/n!$. The derivatives $\Gamma^{(n)}(1)$ can be expressed in terms of Euler's constant and the values of $\zeta(s)$ at positive integers. This property of the c_n is easily derived as follows. Start with the power series expansion for $\psi(x+1) = \Gamma'(x+1)/\Gamma(x+1)$, [1, p. 259],

$$\psi(x+1) = \sum_{n=0}^{\infty} (-1)^{n+1} s_{n+1} x^n, \tag{12}$$

where $s_1 = \gamma$ and $s_n = \zeta(n)$ for $n \geqslant 2$. Equating coefficients of x^n in the identity $\Gamma'(x+1) = \psi(x+1)\Gamma(x+1)$, using (11) and (12), we obtain the recursion formula

$$(n+1)c_{n+1} = \sum_{k=0}^{n} (-1)^{k+1} s_{k+1} c_{n-k} \tag{13}$$

with $c_0 = 1$. (See Nielsen [10, p. 40].)

Equation (13) also leads to a closed form evaluation of the derivatives $\Gamma^{(n)}(1)$ in terms of Euler's constant γ and $\zeta(2), \zeta(3), \ldots$. For example,

$$\Gamma'(1) = -\gamma, \quad \Gamma''(1) = \zeta(2) + \gamma^2, \quad \Gamma'''(1) = -2\zeta(3) - 3\gamma\zeta(2) - \gamma^3,$$
$$\Gamma^{(4)}(1) = 6\zeta(4) + 3\zeta^2(2) + 8\gamma\zeta(3) + 6\gamma^2\zeta(2) + \gamma^4,$$
$$\Gamma^{(5)}(1) = -24\zeta(5) - 20\zeta(2)\zeta(3) - 15\gamma\zeta^2(2) - 30\gamma\zeta(4)$$
$$-20\gamma^2\zeta(3) - 10\gamma^3\zeta(2) - \gamma^5.$$

Jeffery [8] has calculated the first 20 coefficients c_n to 12 decimals. Bourguet [7] later calculated to 16 decimals the first 18 coefficients b_n in the expansion

$$(x+1)\Gamma(x+1) = \sum_{n=0}^{\infty} b_n x^n.$$

This relation implies $b_0 = c_0 = 1$ and

$$b_n = c_n + c_{n-1} \quad \text{for } n \geqslant 1,$$

TABLE 1

n	Stieltjes constants A_n	$c_n = \Gamma^{(n)}(1)/n!$
0	0.5772156649015329	1.000000000000000000
1	0.7281584548367672 (−01)	−0.5772156649015329
2	−0.4845181596436160 (−02)	0.9890559953279726
3	−0.3423057367172240 (−03)	−0.9074790760808863
4	0.9689041939447080 (−04)	0.9817280868344002
5	−0.6611031810842190 (−05)	−0.9819950689031452
6	0.3316240908752770 (−06)	0.9931491146212762
7	0.1046209458447920 (−06)	−0.9960017604424315
8	−0.8733218100273800 (−08)	0.9981056937831289
9	0.9478277782762000 (−10)	−0.9990252676219549
10	0.5658421927608700 (−10)	0.9995156560727774
11	−0.6768689863514000 (−11)	−0.9997565975086013
12	0.3492115936670000 (−12)	0.9998782713151333
13	0.4410424742000000 (−14)	−0.9999390642064443
14	−0.2399786222000000 (−14)	0.9999695177634821
15	0.2167731220000000 (−15)	−0.9999847526993770
16	−0.9544466000000000 (−17)	0.9999923744790732
17	−0.7387700000000000 (−19)	−0.9999961865894733
18	0.4800900000000000 (−19)	0.9999980930811309
		−0.999999046469

TABLE 2

n	a_n	$\zeta^{(n)}(0)$	$\zeta^{(n)}(0)/n!$
0	0.0000000000000000	−0.5000000000000000	−0.5000000000000000
1	0.7286939170039305	−0.9189385332046727	−0.9189385332046727
2	−0.3834560903754670	−2.006356455908585	−1.003178227954292
3	0.5323903060606865	−6.004711166862254	−1.000785194477042
4	−0.4859027759456871	−23.99710318801370	−0.9998792995005709
5	0.5018073423500181	−120.0002329075584	−1.000001940896320
6	−0.4985920362510443	−720.0009368251297	−1.000001301146014
7	0.4998425924690323	−5039.999150176233	−0.9999998313841731
8	−0.4998028591976903	−40320.00023243172	−1.000000005764676
9	0.4999251541081416	−362880.0003305895	−1.000000000911016
10	−0.4999581497598492	−3628799.999456764	−0.9999999998502988
11	0.4999798488252394	−39916800.00037562	−1.000000000009410
12	−0.4999897969263561	−479001600.0000220	−1.000000000000046
13	0.4999949183147713	−6227020799.999629	−0.9999999999999405
14	−0.4999974562188593	−87178291200.00114	−1.000000000000013
15	0.4999987285230217	−1307674368000.008	−1.000000000000006
16	−0.4999993642057035	−20922789888000.15	−1.000000000000007
17	0.4999996821100205	−355687428096002.6	−1.000000000000007
18	−0.49999984106	−6402373705728048.	−1.000000000000008

so we have a simple way of calculating the c_n in (11) recursively from the b_n. The numerical values of the c_n in Table 1 were obtained in this way from Bourguet's values. When rounded off to 12 decimals they agree with Jeffery's results except for c_{10} where Jeffery lists the 12th decimal place as 4 instead of 2.

To relate a_n to the A_n and c_n we write (7) as

$$\Gamma(s+1)\zeta(s+1) = \frac{1}{s} + \sum_{n=0}^{\infty} a_n s^n,$$

then multiply (10) and (11), and equate coefficients to get

$$a_n = c_{n+1} + \sum_{k=0}^{n} A_k c_{n-k}. \tag{14}$$

This gives a closed form evaluation of the a_n in terms of the Stieltjes constants and values of the zeta function at positive integers. The first few values are

$$a_0 = c_1 + A_0 = 0,$$

$$a_1 = \tfrac{1}{2}\zeta(2) - \tfrac{1}{2}\gamma^2 + A_1,$$

$$a_2 = -\tfrac{1}{3}\zeta(3) + \tfrac{1}{3}\gamma^3 - \gamma A_1 + A_2,$$

$$a_3 = \tfrac{1}{4}\zeta(4) + \tfrac{1}{8}\zeta^2(2) - \tfrac{1}{4}\gamma^2\zeta(2) - \tfrac{1}{8}\gamma^4 + \tfrac{1}{2}\zeta(2)A_1 + \tfrac{1}{2}\gamma^2 A_1 - \gamma A_2 + A_3.$$

Numerical values for the a_n are given in Table 2. The calculations were based on (14) using the values for the A_n given by Liang and Todd [9] and the values of c_n listed in Table 1. These values, in turn, were used together with (9) to calculate the derivatives $\zeta^{(n)}(0)$ in Table 2. The numbers in Table 2 reveal that $(-1)^n a_n$ converges to $-\frac{1}{2}$ and that $\zeta^{(n)}(0)/n!$ converges to -1. These facts are easily proved by observing that we have the power series expansions

$$\zeta(1-s) + \frac{1}{s} = \sum_{n=0}^{\infty}\left(\frac{\zeta^{(n)}(0)}{n!} + 1\right)(1-s)^n$$

and

$$\Gamma(s)\zeta(s) + \frac{1}{2s} = \sum_{n=0}^{\infty}\left((-1)^n a_n + \frac{1}{2}\right)(1-s)^n,$$

each of which converges for $s = 0$; so when $s = 0$ the general term of each series tends to 0.

Alternate formulas for $\zeta^{(k)}(s)$ can be obtained from the representation [1, p. 807]

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \sum_{r=1}^{n} \frac{B_{2r}}{2r}\binom{s+2r-2}{2r-1} - \binom{s+2n}{2n+1}\int_1^{\infty} \frac{P_{2n+1}(x)}{x^{s+2n+1}}\,dx \tag{15}$$

which is a consequence of Euler's summation formula. The B_{2r} are Bernoulli numbers and the integral involves the periodic Bernoulli function

$$P_{2n+1}(x) = (-1)^{n+1}\frac{2(2n+1)!}{(2\pi)^{2n+1}} \sum_{k=1}^{\infty} \frac{\sin 2k\pi x}{k^{2n+1}}. \tag{16}$$

The representation for $\zeta(s)$ in (15) is valid in the half-plane $\sigma > -2n$, $n = 1, 2, 3, \ldots,$ and can be rewritten as follows:

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \sum_{r=1}^{n} \frac{B_{2r}}{2r} Q_{2r-1}(s) - Q_{2n+1}(s) I_{2n+1}(s), \tag{17}$$

where $Q_m(s)$ is the binomial coefficient (a polynomial in s of degree m),

$$Q_m(s) = \binom{s+m-1}{m} = \frac{s(s+1)(s+2)\cdots(s+m-1)}{m!}, \tag{18}$$

and

$$I_m(s) = \int_1^{\infty} \frac{P_m(x)}{x^{s+m}}\,dx.$$

The kth derivative of this integral is simply

$$(19)\qquad I_m^{(k)}(s) = (-1)^k \int_1^\infty \frac{P_m(x)(\log x)^k}{x^{s+m}}\,dx,$$

and the kth derivative of $Q_m(s)$ vanishes identically if $k > m$.

By differentiating (17) repeatedly we obtain the following theorem.

THEOREM 4. *In the half-plane $\sigma > -2n$, $n = 1, 2, 3, \ldots,$ we have*

$$(20)\qquad \zeta^{(k)}(s) = \frac{(-1)^k k!}{(s-1)^{k+1}} + \sum_{r=1}^{n} \frac{B_{2r}}{2r} Q_{2r-1}^{(k)}(s) - \sum_{\nu=0}^{k} \binom{k}{\nu} Q_{2n+1}^{(\nu)}(s) I_{2n+1}^{(k-\nu)}(s).$$

For the special case $n = 1$ the sum on r vanishes identically if $k \geqslant 2$, and the sum on ν contains at most four terms since $Q_3^{(\nu)}(s)$ vanishes identically for $\nu > 3$. Thus, for $\sigma > -2$ we have

$$(21)\qquad \zeta'(s) = \frac{-1}{(s-1)^2} + \frac{1}{12} - \frac{s(s+1)(s+2)}{6} I_3'(s) - \frac{3s^2+6s+2}{6} I_3(s),$$

and, for $k \geqslant 2$,

$$(22)\qquad \zeta^{(k)}(s) = \frac{(-1)^k k!}{(s-1)^{k+1}} - \frac{s(s+1)(s+2)}{6} I_3^{(k)}(s) - \frac{k}{6}(3s^2+6s+2) I_3^{(k-1)}(s) - \frac{k(k-1)}{2}(s+1) I_3^{(k-2)}(s) - k(k-1)(k-2) I_3^{(k-3)}(s).$$

(If $k = 2$ the last term on the right of (22) is understood to be zero.)

When $s = 0$ the formulas are even simpler. From (21) we get

$$(23)\qquad \zeta'(0) = -1 + \frac{1}{12} - \frac{1}{3} I_3(0),$$

and for $k \geqslant 2$, we have

$$(24)\qquad \zeta^{(k)}(0) = -k! - \frac{k}{3} I_3^{(k-1)}(0) - \frac{k(k-1)}{2} I_3^{(k-2)}(0) - k(k-1)(k-2) I_3^{(k-3)}(0).$$

These formulas, used in conjunction with Theorem 3, lead to successive closed form evaluations of the integrals $I_3(0)$, $I_3'(0)$, $I_3''(0), \ldots$. For example, using the formulas derived earlier for $\zeta'(0)$ and $\zeta''(0)$, we find

$$I_3(0) = \int_1^\infty \frac{P_3(x)}{x^3}\,dx = -\frac{11}{4} + \frac{3}{2}\log(2\pi) = 0.006815599614018225$$

and

$$I_3'(0) = \int_1^\infty \frac{P_3(x)(-\log x)}{x^3}\,dx = \frac{9}{8} - \frac{9}{4}\log(2\pi) + \frac{3}{4}\log^2(2\pi) - \frac{\pi^2}{16} + \frac{3}{2}a_1 = -0.000688715558150.$$

The same type of analysis can be applied to the Hurwitz zeta function $\zeta(s, a)$, the analytic continuation of the series

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s},$$

where $a > 0$ and $\sigma > 1$. For $\sigma > -1$, Euler's summation formula gives the representation

$$\zeta(s, a) = a^{-s}\left(\frac{1}{2} + \frac{a}{s-1}\right) - s(s+1)\int_0^\infty \frac{\varphi_2(x)}{(x+a)^{s+2}}\,dx, \tag{25}$$

where $\varphi_2(x) = \int_0^x (t - [t] - \frac{1}{2})\,dt$ is periodic with period 1 and satisfies

$$\varphi_2(x) = \tfrac{1}{2}x(x-1) \quad \text{if } 0 \leqslant x \leqslant 1.$$

Differentiating (25) k times we obtain the formula

$$\begin{aligned}(-1)^k \zeta^{(k)}(s, a) = {} & (\log^k a) a^{-s}\left(\frac{1}{2} + \frac{a}{s-1}\right) + k! a^{1-s} \sum_{r=0}^{k-1} \frac{\log^r a}{r!(s-1)^{k-r+1}} \\ & - s(s+1)\int_0^\infty \frac{\varphi_2(x)\log^k(x+a)}{(x+a)^{s+2}}\,dx \\ & + k(2s+1)\int_0^\infty \frac{\varphi_2(x)\log^{k-1}(x+a)}{(x+a)^{s+2}}\,dx \\ & - k(k-1)\int_0^\infty \frac{\varphi_2(x)\log^{k-2}(x+a)}{(x+a)^{s+2}}\,dx.\end{aligned}$$

For $s = 0$ this simplifies to

$$\begin{aligned}\zeta^{(k)}(0, a) = {} & \left(\log^k \frac{1}{a}\right)\left(\frac{1}{2} - a\right) - k! + k! a \sum_{r=k}^{\infty} \frac{\log^r 1/a}{r!} \\ & + (-1)^k k \int_0^\infty \frac{\varphi_2(x)\log^{k-1}(x+a)}{(x+a)^2}\,dx \\ & - (-1)^k k(k-1)\int_0^\infty \frac{\varphi_2(x)\log^{k-2}(x+a)}{(x+a)^2}\,dx.\end{aligned} \tag{26}$$

When $a = 1$ this can be transformed to (24) using integration by parts.

When $a = 1$ and $k = 1$, Eq. (26) reduces to

$$\zeta'(0) = -1 - \int_0^\infty \frac{\varphi_2(x)}{(x+1)^2}\,dx = -1 - \int_1^\infty \frac{\varphi_2(x)}{x^2}\,dx.$$

But, from [2, p. 616], we see that

$$1 + \int_1^\infty \frac{\varphi_2(x)}{x^2}\,dx = \frac{1}{2}\log(2\pi),$$

so we have another derivation of the formula

$$\zeta'(0) = -\tfrac{1}{2}\log(2\pi).$$

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